About Quadratically Connected sequences

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Abstract

The purpose of this article is to find conditions under which two sequences given by linear recurrences of the second order with constant coefficients are quadratically connected. The reason for this was a series of problems in an Math Olympiad style for the solution of which it was necessary in one form or another establish the quadratic connection between them.The text of the article is accompanied by a large number of problems with variational solutions and generalizations

About quadratically p-q generated sequences.

Definition.

Let p, q be real numbers. Sequence $\{x_n\}$ we will call quadratically (p, q)-generated if $x_n = t_n^2, n \in \mathbb{N} \cup \{0\}$ for some sequence $\{t_n\}$, satisfying $t_{n+1} - pt_n + qt_{n-1} = 0, n \in \mathbb{N}$.

(such sequence $\{t_n\}$ we will call sequence-generator).

Theorem 1.

Sequence $\{x_n\}$ can be quadratically (p, q)-generated by some sequence $\{t_n\}$ iff sequence $\{x_n\}$ satisfies to

(1) $x_{n+1} - \left(p^2 - 2q\right) x_n + q^2 x_{n-1} = Mq^n, n \in \mathbb{N}$,

where $x_{0,}\ x_1 \geq 0$ and $M = 2\left(x_1 - p\sqrt{x_1 x_0} + qx_0\right)$

or equivalently

(2) $x_{n+2} - \left(p^2 - q\right) x_{n+1} + \left(qp^2 - q^2\right) x_n - q^3 x_{n-1} = 0, n \in \mathbb{N}$, with $x_{0,} x_1 \geq 0$ and $x_2 = \left(p\sqrt{x_1} - q\sqrt{x_0}\right)^2$.

Proof.

1. Let $x_n = t_n^2, n \in \mathbb{N} \cup \{0\}$ and $t_{n+1} - pt_n + qt_{n-1} = 0, n \in \mathbb{N}$.

Since $t_{n+1} - pt_n + qt_{n-1} = 0$ then $(t_{n+1} + qt_{n-1})^2 = (pt_n)^2 \iff$
$t_{n+1}^2 + 2pt_{n+1}t_{n-1} + q^2 t_{n-1}^2 = p^2 t_n^2$.

From the other hand

$t_{n+1}t_{n-1} - t_n^2 = t_{n-1}(pt_n - qt_{n-1}) - t_n(pt_{n-1} - qt_{n-2}) =$
$q\left(t_n t_{n-2} - t_{n-1}^2\right), n > 1$ implies $t_{n+1}t_{n-1} - t_n^2 = q^{n-1}\left(t_2 t_0 - t_1^2\right) \iff$

$$t_{n+1}t_{n-1} = t_n^2 - q^{n-1}\left(t_1^2 - t_2 t_0\right) = t_n^2 - q^{n-1}\left(x_1 - p\sqrt{x_1 x_0} + qx_0\right) = t_n^2 - \frac{Mq^{n-1}}{2}.$$

Then, $2qt_{n+1}t_{n-1}=2qt_n^2-2Mq^n$ and, therefore,

$$t_{n+1}^2+2qt_{n+1}t_{n-1}+b^2t_{n-1}^2=q^2t_n^2 \iff t_{n+1}^2+2qt_n^2-2Mq^n+q^2t_{n-1}^2=p^2t_n^2 \iff$$

(3) $\quad t_{n+2}^2-\left(p^2-2q\right)t_{n+1}^2+q^2t_n^2=Mq^n, n\in\mathbb{N}.$
Substitution $t_n=\sqrt{x_n}$ in **(3)** gives **(1)**.
Due **(3)** we have
$t_{n+2}^2-\left(p^2-2q\right)t_{n+1}^2+q^2t_n^2-q\left(t_{n+1}^2+2qt_n^2-Mq^n+q^2t_{n-1}^2\right)=0 \iff$
$t_{n+2}^2-\left(p^2-q\right)t_{n+1}^2+\left(qp^2-q^2\right)t_n^2-q^3t_{n-1}^2=0 \implies$ **(2)** and, by **(1)**
$x_2=\left(p^2-2q\right)x_1-q^2x_0+Mq=p^2x_1-2qx_1-q^2x_0+2qx_1-2pq\sqrt{x_1x_0}+$
$2q^2x_0=\left(p\sqrt{x_1}-q\sqrt{x_0}\right)^2.$
2. Let $\{x_n\}$ be sequence defined by **(2)** with $x_0, x_1\geq 0$ and
$x_2=\left(p\sqrt{x_1}-q\sqrt{x_0}\right)^2$, and let $\{t_n\}$ be a sequence which
satisfy to the recurrence
$t_{n+1}-pt_n+qt_{n-1}=0, n\in\mathbb{N}$, where $t_0:=\sqrt{x_0}, t_1:=\sqrt{x_1}$.
We will prove $x_n=t_n^2, n\in\mathbb{N}$ using Math Induction.
Base of Math Induction.
We have $x_0=t_0^2, x_1=t_1^2, x_2=\left(p\sqrt{x_1}-q\sqrt{x_0}\right)^2=(pt_1-qt_0)^2=t_2^2$.
Step of Math Induction.
For any from supposition $x_{n-1}=t_{n-1}^2, x_n=t_n^2$, $x_{n+1}=t_{n+1}^2$ follows
$x_{n+2}=\left(p^2-q\right)x_{n+1}=\left(qp^2-q^2\right)x_n+q^3x_{n-1}=$
$\left(p^2-q\right)t_{n+1}^2-\left(qp^2-q^2\right)t_n^2+q^3t_{n-1}^2=t_{n+2}^2.$ ■
As a corollary note that $\{x_n\}$ can be quadratically $(p,1)$-generated by
some sequence $\{t_n\}$ iff sequence $\{x_n\}$ satisfies to
$x_{n+1}-\left(p^2-2\right)x_n+x_{n-1}=M, n\in\mathbb{N},$
where $x_0,\ x_1\geq 0$ and $M=2\left(x_1-p\sqrt{x_1x_0}+x_0\right)$.

Another example when quadratically (p,q) $-$generated sequence $\{x_n\}$
can be defined by linear second degree recurrence with the constant
in the right hand side we obtain in the case $p=r+1$ and $q=r$, $r\in\mathbb{R}$.
Indeed, in this case
(2) $\iff x_{n+2}-\left(r^2+r+1\right)x_{n+1}+\left(r^3+r^2+r\right)x_n-r^3x_{n-1}=0 \iff$
$x_{n+2}-\left(r^2+r\right)x_{n+1}+r^3x_n-x_{n+1}+\left(r^2+r\right)x_n-r^3x_{n-1}=0 \iff$
$x_{n+2}-\left(r^2+r\right)x_{n+1}+r^3x_n=x_{n+1}-\left(r^2+r\right)x_n+r^3x_{n-1}.$
Hence, $x_{n+1}-\left(r^2+r\right)x_n+r^3x_{n-1}=x_2-\left(r^2+r\right)x_1+r^3x_0=c=const.$
Since $x_2=\left((r+1)\sqrt{x_1}-r\sqrt{x_0}\right)^2$ then
$c=(r+1)^2x_1+r^2x_0-2r(r+1)\sqrt{x_0x_1}-\left(r^2+r\right)x_1+r^3x_0=$
$(r+1)\left(x_1-2r\sqrt{x_0x_1}+r^2x_0\right)=(r+1)\left(\sqrt{x_1}-r\sqrt{x_0}\right)^2.$
Thus $x_{n+1}-\left(r^2+r\right)x_n+r^3x_{n-1}=(r+1)\left(\sqrt{x_1}-r\sqrt{x_0}\right)^2, n\in\mathbb{N}.$
Sequence-generator $\{t_n\}$ satisfy to recurrence
$t_{n+1}-(r+1)t_n+rt_{n-1}=0$
and since $t_{n+1}-rt_n=t_n-rt_{n-1}$ then $t_{n+1}-rt_n=t_1-rt_0$.
Let $d:=t_1-rt_0$ then $t_{n+1}-(r+1)t_n+rt_{n-1}=0 \iff t_{n+1}=rt_n+d$.

(Or, the same result can be obtained by the other way:
Since $t_{n+1}-(r+1)\,t_n+rt_{n-1}=0, n\in\mathbb{N} \iff t_{n+1}=rt_n+d, n\in\mathbb{N}\cup\{0\}$,
where $d:=t_1-rt_0$, then
$(t_{n+1}-d)^2=r^2t_n^2 \iff t_{n+1}^2-r^2t_n^2=2dt_{n+1}-d^2, n\in\mathbb{N}\cup\{0\}$,
and, therefore,
$t_{n+1}^2-r^2t_n^2-r\left(t_n^2-r^2t_{n-1}^2\right)=2drt_{n+1}-d^2-2dr_{n+1}-2dr^2t_n+rd^2=$
$2d\ (t_{n+1}-rt_n)-d^2+rd^2=d^2\,(r+1) \implies$
$t_{n+1}^2-\left(r^2+r\right)t_n^2+r^3t_{n-1}=d^2\,(r+1) \iff$
$x_{n+1}-\left(r^2+r\right)x_n+r^3x_{n-1}=(r+1)\left(\sqrt{x_1}-r\sqrt{x_0}\right)^2, n\in\mathbb{N}.$

Naturally ask a following question:
For which p,q sequence $\{x_n\}$ defined by second degree linear recurrence $x_{n+1}-\mu x_n+\lambda x_{n-1}=\sigma$, where μ,λ,σ some constants is quadratically (p,q)-generated?

For any polynomial $P(x)=x^m+p_1x^{m-1}+p_2x^{n-2}+\ldots+p_{m-1}x+p_m$
and any sequence$\{a_n\}$ let
$L_P(a_n):=a_{n+m-1}+p_2a_{n+m-2}+\ldots+p_{m-1}a_n+a_{n-1}, n\in\mathbb{N}.$
Then for given sequence $\{b_n\}$ recurrence
$a_{n+m-1}+p_2a_{n+m-2}+\ldots+p_{m-1}a_n+a_{n-1}=b_n$
get short notation $L_P(a_n)=b_n, n\in\mathbb{N}$.
Note, the following properties of this notation:
1. $L_{P+Q}(a_n)=L_P(a_n)+L_Q(a_n)$, for two polynomial $P(x), Q(x)$;
2. $L_{cP}(a_n)=cL_P(a_n)$ for constant c and polynomial $P(x)$;
3. if $Q(x)=xP(x)$ then $L_Q(a_n)=L_P(a_{n+1})$.

Lemma.
Let $P(x)=x^3-\alpha x^2+\beta x-\gamma$ such that $P(0)\neq 0$ and $Q(x)=x^2-\lambda x+\mu$.
Then any solution of $L_Q(x_n)=\sigma$ be solution of $L_P(x_n)=0$ iff
$P(x)=(x-1)\,Q(x)$ i.e. $P(1)=0$ and $\alpha=\lambda+1, \beta=\mu+\lambda,\ \gamma=\mu$.

Proof.
Suffieciency.
If $P(x)=(x-1)\,Q(x)=xQ(x)-Q(x)$ then
$L_P(x_n)=L_Q(x_{n+1})-L_Q(x_n)=\sigma-\sigma=0.$
Necessity.
Let $L_Q(x_n)=\sigma \implies L_P(x_n)=0$, where $\{x_n\}\neq 0$. Then, since
$P(x)=(x-1)\,Q(x)+P(1)$ we have
$0=L_P(x_n)=L_Q(x_{n+1})-L_Q(x_n)+P(1)\,x_n=$
$\sigma-\sigma+P(1)\,x_n=P(1)\,x_n \implies P(1)=0.\blacksquare$

Let $P(x):=x^3-\left(p^2-q\right)x^2+\left(p^2q-q^2\right)x-q^3$.
Due to **Theorem 1,** sequence $\{x_n\}$,defined by recurrence
$x_{n+1}-\lambda x_n+\mu x_{n-1}=\sigma$ is quadratically (p,q)-generated by $\{t_n\}$
iff $L_P(x_n)=0$ and $x_0,x_1\geq 0, x_2=\left(p\sqrt{x_1}-q\sqrt{x_0}\right)^2$ and

by **Lemma** it is possible iff
$P(1)=0 \iff 1-\left(p^2-q\right)+\left(p^2q-q^2\right)-q^3=0 \iff 1+q-q^2-q^3-$
$(1-q)p^2=0 \iff (1-q)\left((1+q)^2-p^2\right)=0 \iff \left[\begin{array}{c} q=1 \\ |p|=|q+1| \end{array}\right.$.
And also, by **Lemma** we have $\lambda=p^2-q-1=q\left(p^2-q-q^2\right),\ \mu=q^3$ and $\sigma=x_2-\left(p^2-q-1\right)x_1+q^3x_0$.
If $q=1$ and $p\in\mathbb{R}$ then
$\lambda=p^2-2,\mu=1,\sigma=\left(p\sqrt{x_1}-\sqrt{x_0}\right)^2-\left(p^2-2\right)x_1+x_0=$
$2\left(x_0-p\sqrt{x_0x_1}+x_1\right);$
If $q\in\mathbb{R}$ and $|p|=|q+1|$ then $\lambda=q\left(p^2-q-q^2\right)=q(q+1),\ \mu=q^3,$
$\sigma=\left(p\sqrt{x_1}-q\sqrt{x_0}\right)^2-\left(q^2+q\right)x_1+q^3x_0=(q+1)^2x_1+q^2x_0-2qp\sqrt{x_0x_1}-$
$\left(q^2+q\right)x_1+q^3x_0=\left((q+1)x_1-2qp\sqrt{x_0x_1}+q^2(q+1)x_0\right)=$
$\begin{cases} (q+1)\left(\sqrt{x_1}-q\sqrt{x_0}\right)^2, \text{if } p=q+1 \\ (q+1)\left(\sqrt{x_1}+q\sqrt{x_0}\right)^2, \text{ if } p=-q-1 \end{cases}.$

Thus, we obtain following
Theorem 2. *Only three kinds of sequences $\{x_n\}$ defined by recurrence $x_{n+1}-\lambda x_n+\mu x_{n-1}=\sigma, n\in\mathbb{N}$ can be quadratically (p,q)-generated:*
i. Sequence $\{x_n\}$ defined by

$$x_{n+1}-\left(p^2-2\right)x_n+x_{n-1}=2\left(x_0-p\sqrt{x_0x_1}+x_1\right), n\in\mathbb{N},\ x_0,x_1\geq 0.$$

Then $x_n=t_n^2, n\in\mathbb{N}\cup\{0\}$, where

$$t_{n+1}-pt_n+q_{n-1}=0, n\in\mathbb{N},\ t_0=\sqrt{x_0}, t_1=\sqrt{x_1};$$

ii. Sequence $\{x_n\}$ defined by

$$x_{n+1}-\left(q^2+q\right)x_n+q^3x_{n-1}=(q+1)\left(\sqrt{x_1}-q\sqrt{x_0}\right)^2, n\in\mathbb{N},\ x_0,x_1\geq 0.$$

Then $x_n=t_n^2, n\in\mathbb{N}\cup\{0\}$, where

$$t_{n+1}-(q+1)t_n+q_{n-1}=0, n\in\mathbb{N},\ t_0=\sqrt{x_0}, t_1=\sqrt{x_1};$$

iii. Sequence $\{x_n\}$ defined by

$$x_{n+1}-\left(q^2+q\right)x_n+q^3x_{n-1}=(q+1)\left(\sqrt{x_1}+q\sqrt{x_0}\right)^2, n\in\mathbb{N},\ x_0,x_1\geq 0.$$

Then $x_n=t_n^2, n\in\mathbb{N}\cup\{0\}$, where

$$t_{n+1}+(q+1)t_n+q_{n-1}=0, n\in\mathbb{N},\ t_0=\sqrt{x_0}, t_1=\sqrt{x_1}.$$

Applications.

Problem 1.(O86.MR, Proposed by Brian Bradie, Christopher Newport University, USA).
The sequence $\{a_n\}$ is defined by $a_1=1, a_2=3$ and $a_{n+1}=6a_n-a_{n-1}$ for all $n\geq 1$. Prove that $a_n+(-1)^n$ is a perfect square for all $n\geq 1$.
Solution.
Using $a_1=1, a_2=3$ and $a_{n+1}=6a_n-a_{n-1}$ we can define correctly a_0 as $6a_1-a_2=3$.Let $x_n:=a_n+(-1)^n$ then $x_0=4, x_1=0$ and substitution $a_n=x_n+(-1)^{n+1}$ in $a_{n+1}-6a_n+a_{n-1}=0$ gives us recurrense $x_{n+1}-6x_n+x_{n-1}=8(-1)^{n+1}, n\in\mathbb{N}$ and $x_0=4, x_1=0$.
Easy to see that for $q=-1,\ p=4$ we have
$p^2-2q=6, q^2=1,\ M=x_1-p\sqrt{x_1x_0}+qx_0=-4$ and, accordingly to the **Theorem 1**, $x_n=t_n^2, n\geq 1$, where

$t_{n+1} - 4t_n - t_{n-1} = 0, n \ge 1$ and $t_0 = 2, t_1 = 0$.

Remark (generator of the similar problems).
For any real a, b, p, q where $p^2 \ne 4q$ let sequence $\{a_n\}$ be defined by
$a_{n+1} - \left(p^2 - 2q\right) a_n + q^2 a_{n-1} = 0, n \in \mathbb{N}$
and initial conditions $a_0 = a^2 + c, a_1 = b^2 + cq$,where
$c = \dfrac{2\left(b^2 - pba + qa^2\right)}{4q - p^2}$.
Then $a_n + cq^n = t_n^2, n \in \mathbb{N} \cup \{0\}$,where sequence $\{t_n\}$ satisfy
$t_{n+1} - pt_n + qt_{n-1} = 0, n \in \mathbb{N}$ and $t_0 = a,\ t_1 = b$.
Indeed, easy to see that $x_n := a_n + cq^n$ satisfy
$x_{n+1} - \left(p^2 - 2q\right) x_n + q^2 x_{n-1} = Mq^n, n \in \mathbb{N}$,
where $x_0 = t_0^2, x_1 = t_1^2$, $M = 2\left(t_1^2 - pt_1t_0 + qt_0^2\right)$ and, therefore, accordingly to **Theorem1** sequence $\{x_n\}$ is quadratically (p,q)-generated.

Problem 2.
Let sequence (b_n) satisfy $b_n = \dfrac{b_{n+1} + b_{n-1}}{98}$ for any $n \in \mathbb{N}$ and $b_0 = b_1 = 5$.
Then $\dfrac{b_n + 1}{6}$ is square of integer for any $n \in \mathbb{N} \cup \{0\}$.

Solution1.
Let $x_n := \dfrac{b_n + 1}{6}$ then $x_0 = x_1 = 1$ and by substitution $b_n = 6x_n - 1$
in recurrence $b_{n+1} - 98b_n + b_{n-1} = 0$ we obtain for $\{x_n\}$
following recurrence
$x_{n+1} - 98x_n + x_{n-1} = -16, n \in \mathbb{N}$.
Accordingly to **Theorem 2,case i.** we clame $p^2 - 2 = 98$ and
$2\left(x_0 - p\sqrt{x_0x_1} + x_1\right) = -16 \iff$
$p = \pm 10$ and $2 - p = -8 \iff p = 10$.
Thus $x_n = t_n^2, n \in \mathbb{N} \cup \{0\}$, where $t_{n+1} - 10t_n + t_{n-1} = 0,\ n \in \mathbb{N}$
and $t_0 = t_1 = 1$.

Remark (another solution).
Since b_n positive for all $n \in \mathbb{N} \cup \{0\}$, then we can define $t_n := \sqrt{\dfrac{b_n + 1}{6}}$.

Then $(t_{n+1} + t_{n-1})^2 = \dfrac{b_{n+1} + b_{n-1} + 2 + 2\sqrt{(b_{n+1} + 1)(b_{n-1} + 1)}}{6} =$

$= \dfrac{98b_n + 2 + 2\sqrt{(b_{n+1}b_{n-1} + b_{n+1} + b_{n-1} + 1)}}{6} = \dfrac{98b_n + 2 + 2\sqrt{b_{n+1}b_{n-1} + 98b_n + 1}}{6}$.

Since $b_{n+1}b_{n-1} - b_n^2 = b_2b_0 - b_1^2 = 2400$ then we have

$(t_{n+1} + t_{n-1})^2 = \dfrac{98b_n + 2 + 2\sqrt{b_n^2 + 98b_n + 2401}}{6} = \dfrac{98b_n + 2 + 2\sqrt{(b_n + 49)^2}}{6} =$
$\dfrac{98b_n + 2 + 2(b_n + 49)}{6} = \dfrac{50}{3}(b_n + 1) = (10t_n)^2$ .Thus, $t_{n+1} + t_{n-1} = 10t_n$ and since $t_0 = t_1 = 1$ we conclude that t_n is integer for all $n \in \mathbb{N} \cup \{0\}$.

Generalization. (Generator of this kind of problems)

Theorem 3.
Let sequence $\{t_n\}$ satisfy $t_{n+1} - pt_n + t_{n-1} = 0, n \in \mathbb{N}$ with $t_0 = t_1 = t$, where $p^2 \neq 4$ and sequence $\{a_n\}$ satisfy $a_{n+1} - ra_n + sa_{n-1} = 0, n \in \mathbb{N}$ with $a_0 = a_1 = a$.
Then, $a_n = kt_n^2 + l$ for some k, l iff $s = 1, r = p^2 - 2, k = \dfrac{a(p+2)}{pt^2}, l = -\dfrac{2a}{p}$.
Proof.
Let $x_n := \dfrac{a_n - l}{k}, n \in \mathbb{N} \cup \{0\}$ then $x_0 = x_1 = \dfrac{a-l}{k}$ and, by substitution $a_n = kx_n + l$ in $a_{n+1} - ra_n + sa_{n-1} = 0,\ n \in \mathbb{N}$, we obtain for $\{x_n\}$ following recurrence

$$x_{n+1} - rx_n + sx_{n-1} = \frac{l(r-s-1)}{k},\ n \in \mathbb{N}.$$

By **Theorem 1** $a_n = kt_n^2 + l \iff x_n = t_n^2$ $, n \in \mathbb{N} \cup \{0\} \iff$
$s = 1, r = p^2 - 2, \dfrac{l(r-s-1)}{k} = 2\left(t_1^2 - pt_1t_0 + t_0^2\right) = 2t^2(2-p), \dfrac{a-l}{k} = t^2$.
Thus we have $\dfrac{l\left(p^2-4\right)}{k} = 2t^2(2-p) \iff kt^2 = -\dfrac{l(p+2)}{2}$ and,
since $a - l = kt^2$ then $-\dfrac{l(p+2)}{2} = a - l \iff l = -\dfrac{2a}{p}$ and, therefore,

$$k = -\frac{l(p+2)}{2} = \frac{a(p+2)}{pt^2}.$$

Corollary.
If $a_{n+1} - \left(p^2 - 2\right)a_n + a_{n-1} = 0, n \in \mathbb{N}$ and $a_0 = a_1 = a \neq 0$,
where $a,\ p \in \mathbb{Z}$ and $|p| \neq 2$, then, $\dfrac{t^2pa_n}{a(p+2)} + \dfrac{2t^2}{p+2}$ is square of integer for any $n \in \mathbb{N} \cup \{0\}$.

Problem 3.
Let $a_1 = 1, a_{n+1} = 2a_n + \sqrt{3a_n^2 - 2}, n \in \mathbb{N}$. Prove that all term of this sequence are integers.
Solution 1.
Since $a_n \geq 1 \implies a_{n+1} - 2a_n > 0$ we have $a_{n+1} = 2a_n + \sqrt{3a_n^2 - 2} \iff$
$(a_{n+1} - 2a_n)^2 = 3a_n^2 - 2 \iff a_{n+1}^2 - 4a_{n+1}a_n + 4a_n^2 = 3a_n^2 - 2 \iff$
$a_{n+1}^2 - 4a_{n+1}a_n + a_n^2 = -2 \implies a_n^2 - 4a_na_{n-1} + a_{n-1}^2 = -2$.Hereof
$a_{n+1}^2 - 4a_{n+1}a_n + a_n^2 - \left(a_n^2 - 4a_na_{n-1} + a_{n-1}^2\right) = 0 \iff$
$a_{n+1}^2 - a_{n-1}^2 - 4a_n(a_{n+1} - a_{n-1}) = 0 \iff$
$(a_{n+1} - a_{n-1})(a_{n+1} - 4a_n + a_{n-1}) = 0 \iff$
$a_{n+1} - 4a_n + a_{n-1} = 0,$ since $a_{n+1} > a_{n-1}$.
From the other hand, if $a_{n+1} - 4a_n + a_{n-1} = 0$,then we obtain
$a_{n+1}^2 - 4a_{n+1}a_n + a_n^2 = a_n^2 - 4a_na_{n-1} + a_{n-1}^2 \implies$
$a_n^2 - 4a_na_{n-1} + a_{n-1}^2 = a_2^2 - 4a_2a_1 + a_1^2 = 3^2 - 12 + 1 = -2.$
Solution 2.

Let $t_n := \sqrt{3a_n^2 - 2}$ then $a_{n+1} = 2a_n + t_n$ and from $t_{n+1}^2 = 3a_{n+1}^2 - 2 = 3\left(2a_n + t_n\right)^2 - 2 = 12a_n^2 + 12a_n t_n + 3t_n^2 - 2 = 9a_n^2 + 12a_n t_n + 3t_n^2 + \left(3a_n^2 - 2\right) = \left(3a_n + 2t_n\right)^2$ follows $t_{n+1} = 3a_n + 2t_n$.
From system $\begin{cases} a_{n+1} = 2a_n + t_n \\ t_{n+1} = 3a_n + 2t_n \end{cases}$, by substitution $t_n = a_{n+1} - 2a_n$ in the second recurrence, we obtain
$a_{n+2} - 2a_{n+1} = 3a_n + 2\left(a_{n+1} - 2a_n\right) \iff a_{n+1} - 4a_n + a_{n-1} = 0)$.

Generalization.(Generator of this kind of problems)
Let sequence $\{t_n\}$ defined by $t_{n+1} - 2pt_n + t_{n-1} = 0, n \in \mathbb{N}$,
where $p > 1, t_1 > 0$ and $pt_1 \geq t_0$.
Since $p > 1$ and $pt_1 \geq t_0$ then $t_2 - pt_1 = pt_1 - t_0 \geq 0$.
Using $t_2 \geq pt_1$and $t_1 > 0$ as a base of Math Induction and for any $n \geq 1$ assuming that $t_{n+1} \geq pt_n$ and $t_n > 0$, we obtain $t_{n+1} > 0$ and
$t_{n+2} - pt_{n+1} = pt_{n+1} - t_n \geq \left(p^2 - 1\right) t_n > 0$.
Multiplying $t_{n+1} + t_{n-1} = 2pt_n$ by $t_{n+1} - t_{n-1}$ we obtain
$t_{n+1}^2 - t_{n-1}^2 = 2pt_n t_{n+1} - 2pt_{n-1} t_n \iff$
$t_{n+1}^2 - 2pt_n t_{n+1} + t_n^2 = t_n^2 - 2pt_{n-1} + t_{n-1}^2, n \in \mathbb{N}$.
Hence, $t_{n+1}^2 - 2pt_n t_{n+1} + t_n^2 = c, n \in \mathbb{N}$, where $c = t_1^2 - 2pt_1 t_0 + t_0^2$
and $t_{n+1}^2 - 2pt_n t_{n+1} + t_n^2 = c \iff \left(t_{n+1} - pt_n\right)^2 = \left(p^2 - 1\right) t_n^2 + c \iff$
$t_{n+1} = pt_n + \sqrt{\left(p^2 - 1\right) t_n^2 + c}, n \in \mathbb{N}$,
since $t_{n+1} > pt_n, n \in \mathbb{N}$ and
$\left(p^2 - 1\right) t_n^2 + c > \left(p^2 - 1\right) t_1^2 + t_1^2 - 2pt_1 t_0 + t_0^2 = \left(pt_1 - t_0\right)^2 \geq 0$.
Opposite, let now $\{t_n\}$ be a sequence defined by
$t_{n+1} = pt_n + \sqrt{\left(p^2 - 1\right) t_n^2 + c}, n \in \mathbb{N}$,
where given $p > 1, t_1 > 0$ and c such $\left(p^2 - 1\right) t_1^2 + c \geq 0$.
Then $\{t_n\} = \{t_n\}$ satisfy $t_{n+1} - 2pt_n + t_{n-1} = 0, n \in \mathbb{N}$ with
$t_0 = pt_1 - \sqrt{\left(p^2 - 1\right) t_1^2 + c}$.
Herewith $t_0 \leq pt_1$ and $c = t_1^2 - 2pt_1 t_0 + t_0^2$.
Indeed, then $\{t_n\}$ satisfy to $t_{n+1}^2 - 2pt_n t_{n+1} + t_n^2 = c, n \in \mathbb{N}$, and since $t_{n+1} \geq pt_n$, we obtain $t_{n+2} > t_n, n \in \mathbb{N}$ and
$t_{n+2}^2 - 2pt_{n+1} t_{n+2} + t_{n+1}^2 - \left(t_{n+1}^2 - 2pt_n t_{n+1} + t_n^2\right) = 0 \iff$
$t_{n+2}^2 - t_n^2 = 2pt_{n+1} t_{n+2} - 2pt_n t_{n+1} \iff$
$\left(t_{n+2} - t_n\right)\left(t_{n+2} + t_n - 2pt_{n+1}\right) = 0 \iff t_{n+2} + t_n - 2pt_{n+1} = 0, n \in \mathbb{N}$.
Since $t_2 = pt_1 + \sqrt{\left(p^2 - 1\right) t_1^2 + c}$ then
$t_0 = 2pt_1 - t_2 = pt_1 - \sqrt{\left(p^2 - 1\right) t_1^2 + c} \leq pt_1$ and $c = t_1^2 - 2pt_1 t_0 + t_0^2$.
Thus we obtaine the following theorem and corollary:

Theorem 4.
Let $a > 0,\ p > 1$ and c such that $\left(p^2 - 1\right) a^2 + c \geq 0$.
Then sequence $\{a_n\}$ defined by $a_{n+1} = pa_n + \sqrt{\left(p^2 - 1\right) a_n^2 + c}, n \in \mathbb{N}$ with $a_1 = a$ can be defined by recurrence $a_{n+1} - 2pa_n + a_{n-1} = 0$ with initial conditions $a_1 = a$ and $a_0 = pa - \sqrt{\left(p^2 - 1\right) a^2 + c}$;
Corollary.
Let a be natural number and let p and c be integers such that $p > 1$ and

$\left(p^2-1\right)a^2+c$ is non-negative integer. Then all terms of the sequence $\{a_n\}$, defined by $a_{n+1}=pa_n+\sqrt{\left(p^2-1\right)a_n^2+c}, n\in\mathbb{N}$ with $a_1=a$, are natural numbers.

Remark.
Using idea of the second solution we consider another approach to the general case.
Let $a_{n+1}=pa_n+\sqrt{\left(p^2-1\right)a_n^2+c}, n\in\mathbb{N}$, $a_1>0, p>1$ and c such that $\left(p^2-1\right)a^2+c\geq 0$ then $a_n>0, n\in\mathbb{N}$ and $a_{n+1}^2-2pa_na_{n+1}+a_n^2=c \iff$ $a_n^2-2pa_na_{n+1}=c-a_{n+1}^2$. Denoting $t_n:=\sqrt{\left(p^2-1\right)a_n^2+c}$ we obtain $a_{n+1}=pa_n+t_n,$ and then $t_{n+1}^2:=\left(p^2-1\right)a_{n+1}^2+c=p^2a_{n+1}^2+c-a_{n+1}^2=$ $p^2a_{n+1}^2+a_n^2-2pa_na_{n+1}=\left(pa_{n+1}-a_n\right)^2=\left(\left(p^2-1\right)a_n+pt_n\right)^2 \iff$ $t_{n+1}=\left(p^2-1\right)a_n+pt_n$. From the system $\begin{cases} a_{n+1}=pa_n+t_n \\ t_{n+1}=\left(p^2-1\right)a_n+pt_n \end{cases}$ we obtain $t_{n+2}-2pt_{n+1}+t_n=0$ and $a_{n+2}-2pa_{n+1}+a_n=0$.

★ Problem 4.
Let sequence $\{b_n\}$ defined by $b_{n+1}-6b_n+b_{n-1}=0$ with $b_0=\dfrac{1}{2}, \quad b_1=\dfrac{3}{2}$.
Prove that all terms of sequence $t_n:=\sqrt{2b_n^2-\dfrac{1}{2}}, n\in\mathbb{N}\cup\{0\}$ are integers.

Solution.
Since $b_2=\dfrac{17}{2}$ and $b_{n+1}b_{n-1}-b_n^2=b_{n+1}b_{n-1}-b_n^2=$
$\left(6b_n-b_{n-1}\right)b_{n-1}-b_n\left(6b_{n-1}-b_{n-2}\right)=b_nb_{n-2}-b_{n-1}^2, n\geq 2$
then $b_{n+1}b_{n-1}-b_n^2=b_2b_0-b_1^2=2$.
From the other hand multiplying $b_{n+1}-6b_n+b_{n-1}=0$ by b_{n-1} and using $b_{n+1}b_{n-1}-b_n^2=2$ we obtain
$6b_nb_{n-1}=b_{n+1}b_{n-1}+b_{n-1}^2=b_n^2+b_{n-1}^2+2$.
Let $x_n=2b_n^2-\dfrac{1}{2}$. Then we have
$x_{n+1}=2b_{n+1}^2-\dfrac{1}{2}=2\left(6b_n-b_{n-1}\right)^2-\dfrac{1}{2}=$
$72b_n^2-24b_{n-1}b_n+2b_{n-1}^2-\dfrac{1}{2}=72b_n^2-4\left(b_n^2+b_{n-1}^2+2\right)+2b_{n-1}^2-\dfrac{1}{2}=$
$68b_n^2-2b_{n-1}^2-8-\dfrac{1}{2}=34\left(2b_n^2-\dfrac{1}{2}\right)+17-\left(2b_{n-1}^2+\dfrac{1}{2}\right)-9=$
$34x_n-x_{n-1}+8$.
Thus for $\{x_n\}$ we have $x_{n+1}-34x_n+x_{n-1}=8, n\in\mathbb{N}$, $x_0=0, x_1=4$ and, by Theorem1, $x_n=t_n^2$, $n\in\mathbb{N}\cup\{0\}$, where $\{t_n\}$ defined by $t_{n+1}-6t_n+t_{n-1}=0$ and $t_0=0, t_1=2$.
($q=1, p=6, M=2\left(x_1-6\sqrt{x_1x_0}+x_0\right)=8$).

Problem 5. (S. Harlampiev, Matematika 1989,No.2 ,p.43, Bolgaria)
Sequence $\{a_n\}$ defined as follow

$$a_1 = a_2 = 2,\ a_{n+2} = \frac{2a_{n+1} - 3a_n a_{n+1} + 17a_n - 16}{3a_{n+1} - 4a_n a_{n+1} + 18a_n - 17}, n \in \mathbb{N}.$$

a) Determine a_n as function of n;
b) Prove that all terms of the sequence $\{a_n\}$ can be represented in the form $1 + \dfrac{1}{m^2}$,where $m \in \mathbb{N}$.

Solution.
Using substitution $a_n = b_n + 1$ we obtain $a_{n+2} = \dfrac{2a_{n+1} - 3a_n a_{n+1} + 17a_n - 16}{3a_{n+1} - 4a_n a_{n+1} + 18a_n - 17} \iff$

$a_{n+2} - 1 = \dfrac{(a_n - 1)(a_{n+1} - 1)}{14(a_n - 1) - 4(a_n - 1)(a_{n+1} - 1) - (a_{n+1} - 1)} \iff$

$b_{n+2} = \dfrac{b_{n+1} b_n}{14b_n - 4b_n b_{n+1} - b_{n+1}} \iff \dfrac{1}{b_{n+2}} = \dfrac{14}{b_{n+1}} - \dfrac{1}{b_n} - 4 \iff$

$x_{n+2} - 14x_{n+1} + x_n = -4,$ where $x_n = \dfrac{1}{b_n} = \dfrac{1}{a_n - 1}$ and $x_1 = x_2 = 1.$

Since $x_0 = 14x_1 - x_2 - 4 = 9$ and $14 = p^2 - 2q$ for $p = 4,\ q = 1$ then
$2\left(x_1 - p\sqrt{x_1 x_0} + qx_0\right) = -4$
and, therefore, by Theorem 1 $x_n = t_n^2, n \in \mathbb{N} \cup \{0\}$,where

$$t_{n+1} - 4t_n + t_{n-1} = 0, n \in \mathbb{N} \text{ and } t_1 = t_2 = 1.$$

Problem 6.
The sequence $(x_n)_{\mathbb{N}}$ is given by

$$x_n = \frac{1}{4}\left(\left(2 + \sqrt{3}\right)^{2n-1} + \left(2 - \sqrt{3}\right)^{2n-1}\right), n \in \mathbb{N}.$$

Prove that each x_n equal to the sum of squares of two consecutive integers.

Solution.
First note that $x_n = \dfrac{2 - \sqrt{3}}{4}\left(7 + 4\sqrt{3}\right)^n + \dfrac{2 + \sqrt{3}}{4}\left(7 - 4\sqrt{3}\right)^n$ and, therefore, can be defined by recurrence

(1) $\qquad x_{n+1} - 14x_n + x_{n-1} = 0, n \in \mathbb{N}$

with initial conditions $x_0 = 1, x_1 = 1.$
$(x_2 = 13 = 2^2 + 3^2, x_3 = 14 \cdot 13 - 1 = 9^2 + 10^2).$
We will find a sequence (b_n) of integer numbers such that
$x_n = b_n^2 + (b_n + 1)^2 \iff 2x_n - 1 = (2b_n + 1)^2 \iff y_n = a_n^2$,
where $y_n := 2x_n - 1$ and $a_n := 2b_n + 1.$
By substitution $x_n = \dfrac{y_n + 1}{2}$ in the recurrence **(1)** we obtain

$$\frac{y_{n+1} + 1}{2} - 14 \cdot \frac{y_n + 1}{2} + \frac{y_{n-1} + 1}{2} = 0 \iff y_{n+1} - 14y_n + y_{n-1} - 12 = 0,$$

where $y_0 = y_1 = 1$ and, therefore, $y_2 = 25.$
We will prove that a_n is defined by recurrence

(2) $\qquad a_{n+1} - 4a_n + a_{n-1} = 0,\ n \in \mathbb{N}$

with initial conditions $a_0 = -1, a_1 = 1.$ Obvious that $a_n \in \mathbb{N}.$
Note that
$(a_{n+1} + a_{n-1})^2 = 16a_n^2 \iff a_{n+1}^2 + a_{n-1}^2 + 2a_{n+1}a_{n-1} = 16a_n^2 \iff$
$a_{n+1}^2 + a_{n-1}^2 - 14a_n^2 = 2\left(a_n^2 - a_{n+1}a_{n-1}\right), a_2 = 4 \cdot a_1 - a_0 = 4 + 1 = 5.$
Since $a_{n+1}^2 - a_{n+2}a_n = a_{n+1}(4a_n - a_{n-1}) - (4a_{n+1} - a_n)a_n = a_n^2 - a_{n-1}a_{n+1}$ for any $n \in \mathbb{N}$ then $a_n^2 - a_{n-1}a_{n+1} = a_1^2 - a_0 a_2 = 1 + 5 = 6$ and,

therefore, $a_{n+1}^2+a_{n-1}^2-14a_n^2=12$.
Since $a_1^2=y_1, a_2^2=y_2$ and both sequences $(y_n)_{n\geq 1}$, $\left(a_n^2\right)_{n\geq 1}$ satisfies to the same recurrence then $y_n=a_n^2$ for any $n\in\mathbb{N}$.
By substitution $a_n=2b_n+1$ in the recurrence **(2)** and initial conditions $a_0=-1, a_1=1$ we obtain
$2b_{n+1}+1-4(2b_n+1)+2b_{n-1}+1=0 \iff b_{n+1}-4b_n+b_{n-1}=1, n\in\mathbb{N}$
and $b_0=-1, b_1=0$. And, of course b_n, is integer for any $n\in\mathbb{N}$
(For example $b_2=4\cdot 0-(-1)+1=2, b_3=4\cdot 2-0+1=9, ..$)

Problem 7 (M1174*KVANT)

Sequence of integers $a_1, a_2, ..., a_n, ..$is defined by recurrence

$$a_{n+3}=2a_{n+2}+2a_{n+1}-a_n,\ n\in\mathbb{N}$$

with initial conditions $a_1=1,\ a_2=12,\ a_3=20$.
Prove that for any natural n number $1+4a_na_{n+1}$ is the square of integer number.

Solution.

Since $a_{n+3}-2a_{n+2}-2a_{n+1}+a_n=a_{n+3}-3a_{n+2}+a_{n+1}+$
$a_{n+2}-3a_{n+1}+a_n=0 \iff a_{n+3}-3a_{n+2}+a_{n+1}=(-1)(a_{n+2}-3a_{n+1}+a_n)$
we obtain other equivalent definition of sequence $(a_n)_{\mathbb{N}}$:
(1) $a_{n+2}-3a_{n+1}+a_n=(-1)^{n-1}(a_3-3a_2+a_1)=$
$(-1)^{n-1}(20-36+1)=(-1)^n\,15$.

Remark.

By substitution $a_n=(-1)^n b_n$ in the recurrence **(1)** we obtain the following eqivalent setting of original problem:
Sequence $(b_n)_{\mathbb{N}}$ is defined by recurrence
(2) $b_{n+2}+3b_n+b_n=15$ with $b_1=-1, b_2=12$.
Prove that $1-4b_nb_{n+1}$ is the square of integer number for any $n\in\mathbb{N}$.
But we will use another substitution $a_n=(-1)^n(c_n+3)$ which gives us convenient form for equivalent representation of our problem.
Namely, we have now linear homogenious recurrence

$$c_{n+2}+3c_{n+1}+c_n=0, n\in\mathbb{N} \text{ with } c_1=-4, c_2=9$$

and we have $1+4a_na_{n+1}=1-4b_nb_{n+1}=1-4(c_n+3)(c_{n+1}+3)=$
$-35-12c_n-12c_{n+1}-4c_nc_{n+1}$.
Since $c_0=3$, $c_{n+1}c_{n-1}-c_n^2=c_2c_0-c_1^2=11$ and
$c_{n-1}(c_{n+1}+3c_n+c_{n-1})=0 \iff c_{n-1}c_{n+1}+3c_nc_{n-1}+c_{n-1}^2=0$
we obtain

$$3c_nc_{n-1}=-c_{n-1}^2-c_{n-1}c_{n+1}=-c_{n-1}^2-c_n^2-11.$$

Thus, $1+4a_na_{n+1}=-35-12c_n-12c_{n+1}+8c_nc_{n+1}-12c_nc_{n+1}=$
$-35-12c_n-12c_{n+1}+8c_nc_{n+1}+44+4c_{n+1}^2+4c_n^2=$
$4c_{n+1}^2+4c_n^2+9-12c_n-12c_{n+1}+8c_nc_{n+1}=(2c_{n+1}+2c_n-3)^2$.
Let $t_n:=3-2c_{n+1}-2c_n$,then $1+4a_na_{n+1}=t_n^2$ where t_n satisfy to the recurrence $3-t_{n+1}+3(3-t_n)+3-t_{n-1}=0 \iff$

$$t_{n+1}+3t_n+t_{n-1}=15$$

and $t_0=3-2c_0-2c_1=5, t_1=3-2c_1-2c_2=-7$.

Remark. (Generator of such problems).
Let $\{t_n\}$ satisfy $t_{n+1} - pt_n + t_{n-1} = 0, n \in \mathbb{N}$. Then, using identity
$t_{n+1}t_{n-1} - t_n^2 = t_2t_0 - t_1^2$, we obtain
$t_{n+1}(t_{n+1} - pt_n + t_{n-1}) = 0 \iff pt_nt_{n+1} = t_{n+1}^2 + t_{n+1}t_{n-1} \iff$
$pt_nt_{n+1} = t_{n+1}^2 + t_n^2 - K, n \in \mathbb{N}\cup\{0\}$, where $K := t_1^2 - pt_1t_0 + t_0^2$.
For arbitrary b we have
$(p+2)(t_n+b)(t_{n+1}+b) = (p+2)\left(t_nt_{n+1} + b(t_n+t_{n+1}) + b^2\right) =$
$pt_nt_{n+1} + 2t_nt_{n+1} + b(p+2)(t_n+t_{n+1}) + (p+2)b^2 =$
$t_{n+1}^2 + t_n^2 - K + 2t_nt_{n+1} + b(p+2)(t_n+t_{n+1}) + (p+2)b^2 =$
$(t_n+t_{n+1})^2 + b(p+2)(t_n+t_{n+1}) + (p+2)b^2 - K =$
$\left(t_n + t_{n+1} + \dfrac{b(p+2)}{2}\right)^2 - \dfrac{b^2(p+2)^2}{4} + (p+2)b^2 - K =$
$\left(t_n + t_{n+1} + \dfrac{b(p+2)}{2}\right)^2 - \dfrac{b^2(p^2-4)}{4} - K$. Thus,
$4(p+2)(t_n+b)(t_{n+1}+b) = (2t_n + 2t_{n+1} + b(p+2))^2 - 4K - b^2(p^2-4) \iff$

$$4K + b^2(p^2-4) + 4(p+2)(t_n+b)(t_{n+1}+b) = (2t_n + 2t_{n+1} + b(p+2))^2.$$

Denoting $x_n := t_n + b_n$ we obtain that for $\{x_n\}$ defined by
$x_{n+1} - px_n + x_{n-1} = b(2-p), n \in \mathbb{N}$ and $x_0 = t_0 + b, x_1 = t_1 + b$,
holds $4K + b^2(p^2-4) + 4(p+2)x_nx_{n+1} = (2x_n + 2x_{n+1} + b(p-2))^2$.
For $p = -3, t_0 = 3, t_1 = -4$ and $b = 3$ we obtain
$K = 16 + 3(-12) + 9 = -11$,
$4K + b^2(p^2-4) + 4(p+2)x_nx_{n+1} = 1 - 4x_nx_{n+1}$ and
$(2x_n + 2x_{n+1} + b(p-2))^2 = (2x_n + 2x_{n+1} - 3)^2$.

More generalizations.
1.First we will find recurrence for $\{t_nt_{n+1}\}$.
Since $t_{n+1}^2 - (p^2-2)t_n^2 + t_{n-1}^2 = 2K, n \in \mathbb{N}$ and
$pt_nt_{n+1} = t_{n+1}^2 + t_n^2 - K, n \in \mathbb{N}\cup\{0\}$
then $p\left(t_{n+1}t_{n+2} - (p^2-2)t_nt_{n+1} + t_{n-1}t_n\right) =$
$pt_{n+1}t_{n+2} - (p^2-2)pt_nt_{n+1} + pt_{n-1}t_n =$
$t_{n+2}^2 + t_{n+1}^2 - K - (p^2-2)\left(t_{n+1}^2 + t_n^2 - K\right) + t_n^2 + t_{n-1}^2 - K =$
$\left(t_{n+2}^2 - (p^2-2)t_{n+1}^2 + t_n^2\right) + \left(t_{n+1}^2 - (p^2-2)t_n^2 + t_{n-1}^2\right) - 4K - p^2K = -p^2K \iff$

$$t_{n+1}t_{n+2} - (p^2-2)t_nt_{n+1} + t_{n-1}t_n = -pK.$$

2.Lemma.
Let $\{t_n\}$ satisfy $t_{n+1} - pt_n + t_{n-1} = 0, n \in \mathbb{N}$.
Then for any $p \notin \{0, -1, 2, -2\}$ there is $m \in \mathbb{N}$ such
$(t_{m+3} - t_{m+1})(t_{m+2} - t_m)(t_{m+2} - t_{m+1}) \neq 0$.
Proof.
Consider following cases.
i. There is m such that $t_m = t_{m+1}$ then due to homogeneity of the recurrence $t_{n+1} - pt_n + t_{n-1} = 0, n \in \mathbb{N}$ we can suppose

that $t_m = t_{m+1} = 1$.Also, without loss of generalty, we can assume that $m = 0$.

So, we have $t_0 = t_1 = 1, t_2 = p-1, t_3 = p(p-1) - 1 = p^2 - p - 1$,
Then $t_3 - t_1 = p^2 - p - 1 - 1 = p^2 - p - 2 = (p-2)(p+1) \neq 0$,
$t_2 - t_0 = t_2 - t_1 = p - 2 \neq 0$.
Thus, $(t_{m+3} - t_{m+1})(t_{m+2} - t_m)(t_{m+2} - t_{m+1}) \neq 0$ for $m = 0$;
ii. There is m such that $t_m = t_{m+2}$ then due to homogeneity of the recurrence $t_{n+1} - pt_n + t_{n-1} = 0, n \in \mathbb{N}$ we can suppose that $t_m = t_{m+2} = p$. Also, without loss of generalty we can assume that $m = 0$.
So, we have $t_0 = t_2 = p$.Then, $pt_1 = t_0 + t_2 = 2p \implies t_1 = 2$ and $t_3 = p^2 - 2,\ t_4 = p\left(p^2 - 2\right) - p = p^3 - 3p$.
Hence, $t_4 - t_2 = p^3 - 3p - p = p\left(p^2 - 4\right) \neq 0$,
$t_3 - t_1 = p^2 - 2 - 2 = p^2 - 4 \neq 0, t_3 - t_2 = p^2 - 2 - p = (p-2)(p+1) \neq 0$.
Thus, $(t_{m+3} - t_{m+1})(t_{m+2} - t_m)(t_{m+2} - t_{m+1}) \neq 0$ for $m = 1$.

Theorem.
Let $\{t_n\}$ satisfy $t_{n+1} - pt_n + t_{n-1} = 0, n \in \mathbb{N}$ and let $p \notin \{0, -1, 2, -2\}$.
Then sequences $(t_n t_{n+1})_{n\geq 0}$, $(t_n + t_{n+1})_{n\geq 0}$, $(1)_{n\geq 0}$ are linearly independent, i.e $\alpha t_n t_{n+1} + \beta(t_n + t_{n+1}) + \gamma = 0$ for any $n \in \mathbb{N}\cup\{0\}$ iff $\alpha = \beta = \gamma = 0$.
Proof.
Suppose that there are α, β, γ not all equal to zero such that $\alpha t_n t_{n+1} + \beta(t_n + t_{n+1}) + \gamma = 0$, for any $n \in \mathbb{N}\cup\{0\}$, then (α, β, γ) be solution of the system

$$\begin{cases} \alpha t_n t_{n+1} + \beta(t_n + t_{n+1}) + \gamma = 0 \\ \alpha t_{n+1} t_{n+2} + \beta(t_{n+1} + t_{n+2}) + \gamma = 0 \\ \alpha t_{n+2} t_{n+3} + \beta(t_{n+2} + t_{n+3}) + \gamma = 0 \end{cases}$$

for any $n \in \mathbb{N}\cup\{0\}$.
Note that

$$\det\begin{pmatrix} t_n t_{n+1} & t_n + t_{n+1} & 1 \\ t_{n+1} t_{n+2} & t_{n+1} + t_{n+2} & 1 \\ t_{n+2} t_{n+3} & t_{n+2} + t_{n+3} & 1 \end{pmatrix} =$$

$$(t_{n+3} - t_{n+1})(t_{n+2} - t_n)\det\begin{pmatrix} t_n t_{n+1} & t_n + t_{n+1} & 1 \\ t_{n+1} & 1 & 0 \\ t_{n+2} & 1 & 0 \end{pmatrix} =$$

$$(t_{n+3} - t_{n+1})(t_{n+2} - t_n)(t_{n+2} - t_{n+1}).$$

Since by **Lemma** always exist at least one m such that $(t_{m+3} - t_{m+1})(t_{m+2} - t_m)(t_{m+2} - t_{m+1}) \neq 0$ then from system

$$\begin{cases} \alpha t_m t_{m+1} + \beta(t_m + t_{m+1}) + \gamma = 0 \\ \alpha t_{m+1} t_{m+2} + \beta(t_{m+1} + t_{m+2}) + \gamma = 0 \\ \alpha t_{m+2} t_{m+3} + \beta(t_{m+2} + t_{m+3}) + \gamma = 0 \end{cases} \quad \text{follows } \alpha = \beta = \gamma = 0.$$

Let $f(x, y) := a\left(x^2 + y^2\right) + bxy + c(x + y) + d$ then $f(t_n, t_{n+1}) =$
$a\left(t_n^2 + t_{n+1}^2\right) + bt_n t_{n+1} + c(t_n + t_{n+1}) + d =$
$a(pt_n t_{n+1} + K) + bt_n t_{n+1} + c(t_n + t_{n+1}) + d =$
$(ap + b)t_n t_{n+1} + c(t_n + t_{n+1}) + d + aK$.

Theorem 5.
Let sequence $\{t_n\}$ satisfy $t_{n+1}-pt_n+t_{n-1}=0, n\in\mathbb{N}$ where $p\notin\{0,-1,2,-2\}$ and $t_0^2+t_1^2\neq 0$ then $f(t_n,t_{n+1})=(\alpha t_n+\alpha t_{n+1}+\beta)^2, n\in\mathbb{N}\cup\{0\}$ for some α and β iff

$$\begin{cases} ap+b=\alpha^2(2+p) \\ c=2\alpha\beta \\ d+aK=\beta^2+\alpha^2K \end{cases}$$

Proof.
Since
$(\alpha t_n+\alpha t_{n+1}+\beta)^2=\alpha^2\left(t_n^2+t_{n+1}^2\right)+2\alpha^2t_nt_{n+1}+2\alpha\beta(t_n+t_{n+1})+\beta^2=$
$\alpha^2(pt_nt_{n+1}+K)+2\alpha^2t_nt_{n+1}+2\alpha\beta(t_n+t_{n+1})+\beta^2=$
$\alpha^2(2+p)t_nt_{n+1}+2\alpha\beta(t_n+t_{n+1})+\beta^2+\alpha^2K$ then by Theorem
$f(t_n,t_{n+1})=(\alpha t_n+\alpha t_{n+1}+\beta)^2, n\in\mathbb{N}\cup\{0\}\iff$
$\left(ap+b-\alpha^2(2+p)\right)t_nt_{n+1}+(c-2\alpha\beta)(t_n+t_{n+1})+$
$d+aK-\beta^2-\alpha^2K=0,\ n\in\mathbb{N}\cup\{0\}$
iff $ap+b=\alpha^2(2+p),\ c=2\alpha\beta$ and $d+aK=\beta^2+\alpha^2K.$

More general analysis associated with problem 7.
First, we will find recurrence for sequence $\left(a_n^2\right)$ where (a_n) be defined by recurrence

$$a_{n+1}-2pa_n+a_{n-1}=0, n\in\mathbb{N}.$$

Since $4p^2a_n^2=a_{n+1}^2+a_{n-1}^2+2a_{n+1}a_{n-1}$ and
$M:=a_2a_0-a_1^2=a_{n+1}a_{n-1}-a_n^2, n\in\mathbb{N}$
we obtain $4p^2a_n^2=a_{n+1}^2+a_{n-1}^2+2a_n^2+2M\iff$
$a_{n+1}^2-2\left(2p^2-1\right)a_n^2+a_{n-1}^2=-2M.$
Second, we will find recurrence for sequence (a_na_{n+1})
Multiplying both sides of recurrence $a_{n+2}-2pa_{n+1}+a_n=0$ by a_n
we obtain $a_{n+2}a_n-2pa_{n+1}a_n+a_n^2=0\iff 2pa_{n+1}a_n=a_{n+1}^2+a_n^2+M.$
Hence, $2p\left(a_{n+2}a_{n+1}-2\left(2p^2-1\right)a_{n+1}a_n+a_na_{n-1}\right)=$
$a_{n+2}^2+a_{n+1}^2+M-2\left(2p^2-1\right)a_{n+1}^2-2\left(2p^2-1\right)a_n^2-4p^2M+$
$2M+a_n^2+a_{n-1}^2+M=a_{n+2}^2-2\left(2p^2-1\right)a_{n+1}^2+a_n^2+2M+a_{n+1}^2-$
$2\left(2p^2-1\right)a_n^2+a_{n-1}^2+2M-4p^2M=-4p^2M\iff$

$$a_{n+2}a_{n+1}-2\left(2p^2-1\right)a_{n+1}a_n+a_na_{n-1}=-2pM.$$

This is interesting recurrence, but more important now correlation

$$2pa_{n+1}a_n=a_{n+1}^2+a_n^2+M,$$

because in the first, it show the way how construct problems like **Problem 7** and in the second it is the base for the following generalization, namely we will prove that for any natural m holds representation

$$a_na_{n+m}=\alpha_ma_n^2+\beta_ma_{n+1}^2+\gamma_m.$$

1. We start from the special linear combination of a_n and a_{n+1}, namely let α,β be arbitrary real number then
$(\alpha a_{n+1}+\alpha a_n+\beta)^2=\alpha^2\left(a_{n+1}^2+a_n^2+M\right)+\beta^2-\alpha^2M+$
$2\alpha^2a_{n+1}a_n+2\alpha\beta a_{n+1}+2\alpha\beta a_n=$

$2p\alpha^2 a_{n+1}a_n + 2\alpha^2 a_{n+1}a_n + 2\alpha\beta a_{n+1} + 2\alpha\beta a_n + \beta^2 - \alpha^2 M =$
$2\alpha^2(p+1)a_{n+1}a_n + 2\alpha\beta a_{n+1} + 2\alpha\beta a_n + \beta^2 - \alpha^2 M.$
So, for given $p, \alpha, \beta, a, b,$ if $a_{n+1} - 2pa_n + a_{n-1} = 0, n \in \mathbb{N}$
with $a_0 = a, a_1 = b$ then $M = 2pab - a_2 - b^2$ and

$$2\alpha^2(p+1)a_{n+1}a_n + 2\alpha\beta a_{n+1} + 2\alpha\beta a_n + \beta^2 - \alpha^2 M = (\alpha a_{n+1} + \alpha a_n + \beta)^2.$$

Note,that for $\alpha = 2, \beta = -3, p = -\dfrac{3}{2}, a = 3, b = -4$ we obtain
$-4a_{n+1}a_n - 12a_{n+1} - 12a_n + 9 - 4(36 - 9 - 16) =$
$-4a_{n+1}a_n - 12a_{n+1} - 12a_n - 35 = (2a_{n+1} + 2a_n - 3)^2.$
For some suitable constant $\delta, \eta, \theta, \zeta$ we can consider quadratic form
$\delta a_{n+1}^2 + \eta a_{n+1}a_n + \delta a_n^2 + \theta a_{n+1} + \theta a_n + \zeta$ which with using identity
$2pa_{n+1}a_n = a_{n+1}^2 + a_n^2 + M$ can be transformed to the $(\alpha a_{n+1} + \alpha a_n + \beta)^2$.
It should be constant $\delta, \eta, \theta, \zeta$ such that
$\eta + 2p\delta = 2\alpha^2(p+1),\ \theta = 2\alpha\beta, \zeta - \delta M = \beta^2 - \alpha^2 M.$
Other, more difficult problem can be constructed if we use sum of
two squares $(\alpha a_{n+1} + \alpha a_n + \beta)^2 + (\gamma a_{n+1} + \delta a_n + \lambda)^2$.
2. Since $\alpha_{m+1}a_n^2 + \beta_{m+1}a_{n+1}^2 + \gamma_{m+1} = a_n a_{n+m+1} =$
$2pa_n a_{n+m} - a_n a_{n+m-1} =$
$2p\left(\alpha_m a_n^2 + \beta_m a_{n+1}^2 + \gamma_m\right) - \left(\alpha_{m-1}a_n^2 + \beta_{m-1}a_{n+1}^2 + \gamma_{m-1}\right)$
we can see that α_m, β_m and γ_m satisfy to the same recurrence
$x_{m+1} - 2px_m + x_{m-1} = 0$ but have different initial conditions.
From $a_n^2 = 1 \cdot a_n^2 + 0 \cdot a_{n+1}^2 + 0$ we obtain $\alpha_0 = 1, \beta_0 = 0$ and $\gamma_0 = 0$.
From $a_n a_{n+1} = \dfrac{1}{2p}a_n^2 + \dfrac{1}{2p}a_{n+1}^2 + \dfrac{M}{2p}$ we obtain
$\alpha_1 = \dfrac{1}{2p}, \beta_1 = \dfrac{1}{2p}$ and $\gamma_1 = \dfrac{M}{2p}$.
For example $\alpha_2 = 2p \cdot \dfrac{1}{2p} - 1 = 0, \beta_2 = 2p \cdot \dfrac{1}{2p} - 0 = 1, \gamma_2 = 2p \cdot \dfrac{M}{2p} - 0 = M,$
thus $a_n a_{n+2} = a_{n+1}^2 + M$;
$\alpha_3 = 2p \cdot 0 - \dfrac{1}{2p} = -\dfrac{1}{2p}, \beta_3 = 2p \cdot 1 - \dfrac{1}{2p} = \dfrac{4p^2 - 1}{2p}, \gamma_3 = 2p \cdot M - \dfrac{M}{2p} =$
$\dfrac{M(4p^2 - 1)}{2p}$,thus $a_n a_{n+3} = -\dfrac{a_n^2}{2p} + \dfrac{a_{n+1}^2(4p^2 - 1)}{2p} + \dfrac{M(4p^2 - 1)}{2p}.$
Using representation $a_n a_{n+m} = \alpha_m a_n^2 + \beta_m a_{n+1}^2 + \gamma_m$ we obtain
$a_{n+1}a_{n+m+1} - 2(2p^2 - 1)a_n a_{n+m} + a_{n-1}a_{n+m-1} =$
$\alpha_m\left(a_{n+1}^2 - 2(2p^2 - 1)a_n^2 + a_{n-1}^2\right) + \beta_m\left(a_{n+2}^2 - 2(2p^2 - 1)a_{n+1}^2 + a_n^2\right) + \gamma_m\left(1 - 2(2p^2 - 1) + 1\right) =$
$-2M(\alpha_m + \beta_m) + 4(1 - p^2)\gamma_m.$
Thus, for any fixed $m \geq 0$ we have the following recurrence for $a_n a_{n+m}$:
$a_{n+1}a_{n+m+1} - 2(2p^2 - 1)a_n a_{n+m} + a_{n-1}a_{n+m-1} = -2M(\alpha_m + \beta_m) + 4(1 - p^2)\gamma_m.$